

Hénon's generating solutions and the structure of periodic orbits families of the restricted three-body problem

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Abstract

We propose a survey of Michel Hénon works devoted to studying periodic solutions of the well-known celestial mechanics problem – restricted three-body problem. The description of the main results obtained by Hénon is given in comparison with results of russian mathematician Alexander Bruno. Finally, a survey of Hénon's works on Hill problem is given as well, and, more over, the authors propose some generalization of Hill problem that makes possible to provide the description of its families of periodic orbits in the form of common network.

1 Introduction

There are two problems in theoretical mechanics, which influence on developing of many branches of mathematics and mechanics can not be underestimated:

- the problem of rotation of rigid body with one fixed point;
- the restricted three-body problem (RTBP).

Planar circular RTBP is an example of rather simple but non-integrable Hamiltonian system with two degrees of freedom, which possesses all the richness of its phase flow behavior. The most interesting objects of this problem are periodic orbits and their families. On our opinion the following quote from [1, Ch. III, § 36] gives us an explanation of importance of periodic orbits.

...ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable.

Families of periodic solutions of dynamic systems with two degrees of freedom form a frame of system behavior. That is why a lot of methods of periodic orbits calculation were developed, but overwhelming majority of these methods are designed for so called regular cases, when solution of unperturbed system moves far away of singular points of perturbing function. The singular case is much more complicated and regular case methods do not applicable. Michel Hénon provided an approach based on a concept of so called “generating solution”. This approach together with the highest level of numerical computations allowed to reach the understanding of structure of the main families of periodic solutions of the planar circular RTBP.

The first Hénon’s paper [2] devoted to periodic solutions of the RTBP was published in 1965 and the last his paper [3] was published in 2005. During this period M. Hénon consequently developed and used an approach based on generating solution technique for studying periodic orbits of the second species or *orbits with consecutive collisions* as they were called by Poincaré. Almost at the same time another researcher – professor A. D. Bruno – provided his own studying of the RTBP’s periodic solutions. They independently obtained the structure of generating solutions of the RTBP and from here mutually interacted with each other during about 30 years. In presented contribution we want to show that there was such a kind of scientific competition between Michel Hénon and Alexander Bruno.

2 Survey of M. Hénon’s works on the RTBP

As Hénon mentioned in the Preface to the book [4] his work over generating solutions of the RTBP lasted about thirty years. During this period he interrupted his study and then came back again. May be it is possible to divide the thirty years long interval of the RTBP’s orbits investigation into three main blocks.

2.1 First block of Hénon’s study

During 1965-70 Hénon presented some papers devoted to numerical exploration of periodic solutions of the RTBP. Symmetric orbits for ($\mu = 1/2$) were investigated.

Let us recall the setting of the RTBP.

Two massive pointlike bodies M_1 and M_2 move in a plane under the influence of Newton’s gravitational law around their center of mass along circular orbits. In the same plane of motion there is another one massless body M_3 which motion one want to study. The units of distance, mass and time are selected in a such way that the distance between M_1 and M_2 , total mass of these bodies, gravitational constant are equal to unity and the period of revolution is equal to 2π . In synodical i.e. uniformly rotating frame body M_1 is situated in the origin and the position of the body M_2 is point with coordinates (1,0). The only parameter of the problem is so called *mass parameter* μ – the mass of

the second body. The Hamiltonian function of the RTBP in rotating frame is following

$$H = \frac{1}{2} (y_1^2 + y_2^2) + x_2 y_1 - x_1 y_2 - \frac{1}{r_1} + \mu \left(\frac{1}{r_1} + x_1 - \frac{1}{r_2} \right), \quad (1)$$

where $r_1^2 = x_1^2 + x_2^2$, $r_2^2 = (x_1 - 1)^2 + x_2^2$.

From the practical point of view the most interesting case for investigation is when parameter μ is rather small but particular case $\mu = 1/2$ – so called Copenhagen case – attracted its attention and Hénon numerically studied this case as well. We guess that this case was selected by Hénon for investigation due to presence of great computation work provided by the group of Strömberg in 1930-th. So he had possibility to compare his own numerical results with those that were calculated earlier and by different methods.

Two papers [2, 5] demonstrated the highest level of numerical investigation of the problem. In brief the essential properties of Hénon's work are following.

- Accurate and thorough numerical computations in regular coordinates using global Thiele regularization (see [6]).
- Poincaré map technique was used for bifurcation analysis of periodic solutions.
- Recomputation of earlier found orbits and investigation of six new families of periodic solutions with asymptotic behavior.

Very interesting class of orbits with asymptotic behavior were studied. These orbits exist for definite values of parameter μ , namely, when the eigenvalues of matrix Hess H computed at the triangular Lagrange libration points L_4 or L_5 are complex with non-zero real and imaginary parts.

The next paper [7] from the first block presented the main ideas for studying second species periodic solutions with the help of generating solution technique. Let us give some definition.

Definition 1 *Let at small parameter value $\mu > 0$ there exists a periodic solution $z(t, \mu)$ to canonical system defined by Hamiltonian*

$$H(z) = H_0(z) + \mu R(z),$$

*and it could be smoothly continued over μ . Then its limit at $\mu \rightarrow 0$ (if it does exist) is called **generating solution**. Here $z = (x, y)$, x is a vector of coordinates and y is a vector of canonically conjugated momenta.*

We assume that the system of canonical equations has a periodic solution $z(t, z_0)$ with period T : $z(T, z_0) = z_0$ for definite value of small parameter μ . Let ρ'_μ and ρ''_μ are the minimal and the maximal distances correspondingly from the body M_2 to the orbit $x(t, z_0)$. There are three possible ways of continuation of periodic solution $z(t, z_0)$ while $\mu \rightarrow 0$:

1. If $\lim_{\mu \rightarrow 0} \rho'_\mu > 0$ then we get generating solution of the *first species*; such solution is regular and can be found by the normal form method (see [8, Ch. II, VII]).
2. If $\lim_{\mu \rightarrow 0} \rho'_\mu = 0$ and $\lim_{\mu \rightarrow \infty} \rho''_\mu > 0$ then we get generating solution of the *second species*.
3. If $\lim_{\mu \rightarrow 0} \rho'_\mu = \lim_{\mu \rightarrow \infty} \rho''_\mu = 0$ then we get generating solution of the *third species*.

If generating solution is not regular then it may consist of solutions of special form called *arc-solution*. The arc-solutions start and finish at the singular points of the Hamiltonian H . So, generating solution of the second species can be constructed from the arc-solutions.

M. Hénon found out two types of arc-solutions called *S*-arcs and *T*-arcs. Let consider the case $\mu = 0$ in sidereal i.e. inertial frame. The bodies M_2 and M_3 execute Keplerian motion, M_2 uniformly moves along the unit circle and M_3 moves along the ellipse (see Fig. 1).

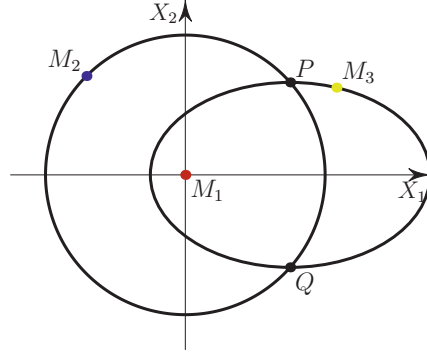


Figure 1: Orbits in sidereal frame for the case $\mu = 0$.

The main difference between the Kepler problem and this case is collisions between bodies M_2 and M_3 . The position of the bodies are defined as follows:

$$\begin{aligned}
 M_2 : \quad x_1 &= \cos \tau, & x_2 &= \sin \tau, \\
 M_3 : \quad x_1 &= \varepsilon a (\varepsilon'' \cos \eta - e), & x_2 &= \varepsilon \varepsilon' a \sqrt{1 - e^2} \sin \eta, \quad t = a^{3/2} (\eta - \varepsilon'' e \sin \eta),
 \end{aligned}$$

where η is true anomaly, a and e are the semi-major axis and the eccentricity of the ellipse, parameters ε , ε' , ε'' can take values ± 1 , defining the direction of the motion and position of the pericenter of the ellipse.

Two cases of collisions are possible.

1. The collision takes place at the same point P at the moment t_1 . Therefore, the sidereal periods of motion are commensurable and such arcs were denoted by T .
2. The first collision take place at the point P at the moment t_1 and the second collision takes place at the symmetrical point Q at the moment t_2 . The set of symmetric arc-solutions was denoted by Hénon as S -arcs.

Hénon obtained an equation relating time τ and true anomaly η only

$$\sqrt{1 - \varepsilon\varepsilon'' \cos \tau \cos \eta} [\eta(1 - \varepsilon\varepsilon'' \cos \tau \cos \eta) - \sin \eta(\cos \eta - \varepsilon\varepsilon'' \cos \tau)] - \tau |\sin \eta|^3 = 0 \quad (2)$$

and numerically solved it for $0 < \tau < 5.5\pi$ and $0 < \eta < 6.5\pi$. He found countable many analytic solutions of the equation (2) grouped into three sequences A_i , B_j and C_{ik} , where indices $i \in \mathbb{N}_0$, $j, k \in \mathbb{N}$. The corresponding curves are shown in Fig. 2 taken from [4]. Hénon also compute critical orbits of these fam-

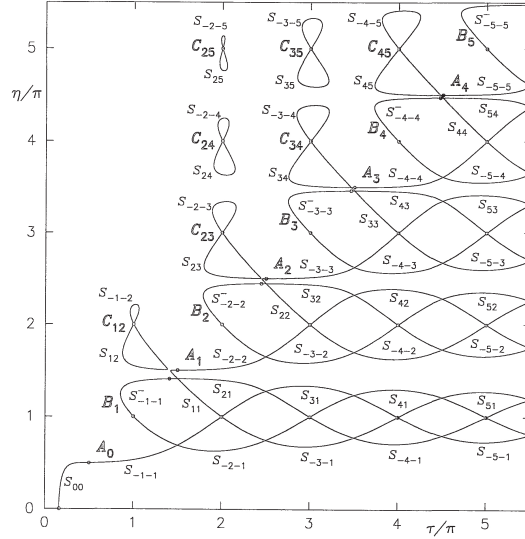


Figure 2: Characteristic of arc families in (τ, η) plane.

ilies. This was the first step to understanding the structure of the generating solutions of the RTBP's periodic orbits.

Last two papers [9, 10] written at the same time were devoted to studying Hill problem which is the limiting case of the RTBP. We consider Hénon's work on Hill problem in the next section.

2.2 Bruno results on the RTBP

Since the beginning of 1970th Alexander Bruno started his own program of investigation of periodic solutions of the RTBP. His candidate and doctoral theses were devoted to the problem of convergence of normal forms of ordinary differential equations and their systems. So, using this approach A. Bruno independently obtained the description of generating solutions of periodic orbits of the RTBP. He published his results in Soviet scientific journals and preprints of Keldysh Institute of Applied Mathematics in Russian. As Hénon wrote in the Preface to his book [4] “I was so impressed by his work that ... I translated two of his papers into English ...”. Finally Bruno’s papers have been collected into book published in Russian in 1990 and translated into English in 1994. This book provided theoretical background of investigation of Hamiltonian systems with the method of normal form and proposed a survey of generating solutions of the first species (regular case) and of the second species (singular case). Bruno also studied bifurcations of generating solutions of the first species but the same analysis for generating solutions of the second species was not provided yet.

The direct face-to-face encounter between Michel Hénon and Alexander Bruno took place at University of Nice at August, 1990, when Bruno was invited as visit professor.

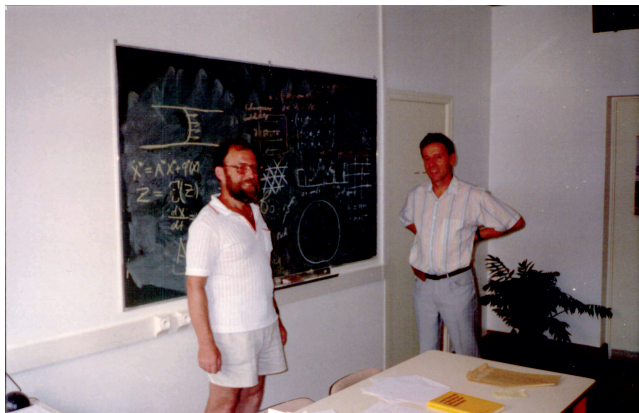


Figure 3: Michel Hénon (right) and Alexander Bruno (left) in University of Nice, August, 1990

2.3 Hénon’s books

The second period of Hénon’s investigation of the RTBP periodic solutions was summarized in his two monographs published in Springer. The first one named “Generating Families in Restricted Three-Body Problem” was based on the sequel of his papers and also papers written in collaboration with Pierre Guillaume, Donald Hitzl. He also used the results obtained by Lawrence Perko

and especially by Alexander Bruno. The main achievements of this book were precise description of structure of second species generating solutions, their bifurcation analysis and comparison with so called natural families which were computed by Broucke in 1968 in the Earth-Moon case ($\mu = 0.012155$) and by Bruno in 1993, 1996 in the Sun-Jupiter case ($\mu = 0.00095388$). As concerning the structure of the generating solutions the main proposition was presented in Chapter 4 of the book:

Proposition 1 (Proposition 4.3.2 from [4]) *An ordinary generating orbit of the second species cannot contain two identical T -arcs of type 1 in succession.*

The background of this Proposition 1 was provided rather heuristic, so, to strictly complete generating solutions investigation and to give quantitative analysis of bifurcations, Hénon decided to write one more book. As he wrote at the beginning of Chapter 15 in [11] “After the present monograph was completed ... there appear a book by Bruno (1998, 2000)”. Indeed, in 1998 Alexander Bruno published his new book “Power Geometry in Algebraic and Differential Equations” in Russian, which was translated into English and published in 2000 by Elsevier [12]. The book of Bruno proposed the method for quantitative analysis of solutions near singularities and therefore gave an instrument for bifurcation analysis of generating solutions. M. Hénon considerably rewrote his second volume and published its new version with strict background of bifurcation analysis of generating families in 2001.

But competition between Hénon and Bruno did not finish. During 2005–2012 A. Bruno mainly together with V. Varin published a sequel of papers where many families of periodic solutions were computed and described in the case $\mu \rightarrow 0$. Here is an abstraction from Alexander Bruno paper list of last papers, which are related to the problem of computation of families of periodic orbits with the help of their generating solutions.

- 65 On periodic flybys of the moon. *Celestial Mechanics* 24:3 (1981) 255-268.
- 258 Periodic solutions to a Hamiltonian system // *Cosmic Research* 44:3 (2006) 245-257.
- 271 On families of periodic solutions of the restricted three-body problem (with V.P. Varin) // *Celestial Mechanics and Dynamical Astronomy*, 2006, v. 95, p. 27-54.
- 283 Periodic solutions of the restricted three-body problem for small mass ratio (with V.P. Varin) // *J. Appl. Math. Mech.* 71:6 (2007) 933-960.
- 291 On families of periodic solutions of the restricted three-body problem (with V.P. Varin) // *Solar System Research* 42:2 (2008) 158-180.
- 304 Family h of periodic solutions of the restricted problem for small μ (with V.P. Varin) // *Solar System Research* 43:1 (2009) 2-25.
- 305 Families c and i of periodic solutions of the restricted problem for $\mu = 5 \cdot 10^{-5}$ (with V.P. Varin) // *Solar System Research* 43:1 (2009) 26-40.

- 306 Family h of periodic solutions of the restricted problem for big μ (with V.P. Varin) // Solar System Research 43:2 (2009) 158-177.
- 307 Closed families of periodic solutions of the restricted problem (with V.P. Varin) // Solar System Research 43:3 (2009) 253-276.
- 337 On asteroid distribution (with V.P. Varin) // Solar System Research 45:4 (2011) 323-329.
- 362 Periodic solutions of the restricted three body problem for small μ and the motion of small bodies of the Solar system (with V.P. Varin) // Astronomical and Astrophysical Transactions (AApTr), 2012, vol. 27, Issue 3, pp. 479-488.

3 Hénon's works on Hill problem

Our first acquaintance with Hénon's papers took place in 1995 when we were offered to read two papers [9, 10], devoted to Hill problem. From that moment studying families of periodic solutions of Hamiltonian systems in general and of Hill problem in particular became the main field of our scientific researches. At the end of 2004 when the second author was finishing her candidate thesis we decided to write Michel Hénon a message for the purpose to get an independent evaluation of the level of fulfilled work. He asked us to send him our last results concerning non-symmetric periodic solutions of Hill problem because at the very moment he was finishing his paper [3]. Later before the thesis defense the second author sent to Hénon abstract of her thesis and he appreciated of the obtained results.

Planar Hill problem is a limit case of the planar RTBP. The equations of motion of Hill problem describe the motion of massless body near the smaller active mass (the Earth), while the last moves around the bigger active mass (the Sun) along the circular orbit.

Hill problem was originally proposed by American astronomer George Hill for the Moon motion theory. He had obtained approximate periodic solutions in the form of trigonometric series, which then were used by him as intermediate solutions for his theory of the Moon's motion. For more details about the role of the Hill problem in constructing the theory of the Moon's motion see [13].

Hill problem is an nonintegrable Hamiltonian problem and it possesses the infinite number of periodic solution. Some of these solutions can be used for continuation into periodic orbits of the RTBP or into periodic orbits of generic three-body problem.

M. Hénon made a major contribution to investigations of periodic solutions to planar Hill problem. In papers [9, 10, 14] he summarized the previous results obtained numerically, gave a description of main properties of periodic solutions, and also investigated their vertical stability. In papers [15, 3] M. Hénon applied the method of generating solutions, used in his book [4] in the context of planar RTBP, to studying periodic solutions of planar Hill problem. He also

described new families of periodic solutions. In many aspects this paper is a continuation and development of ideas and methods described in [15, 4] (and also in book [8] by A.D. Bruno) for seeking periodic solutions based on their generating solutions. However, M. Hénon in the above papers restricted himself to investigation of only families of those periodic solutions which included orbits with a global multiplicity of three and less. Moreover, a major part of families described in paper [15] were obtained by him using the “brute-force” method. This method turned out to be unsuitable for seeking families of asymmetric solutions. Therefore, all families described in paper [3] were found as a result of bifurcation with loss of symmetry of earlier studied families of symmetric periodic solutions.

Hill problem has a lot of applications to analyzing the orbit evolution of natural satellites, to design orbits for space missions, and to studying dynamics of star clusters.

3.1 Second species generating solutions

In [9] Hénon proposed to look for generating solutions in the set of arc-orbits which start and finish at the origin, which is the only singular point of the Hamiltonian of Hill problem, which is following:

$$H(x, y) = \frac{1}{2} (y_1^2 + y_2^2) + x_2 y_1 - x_1 y_2 - x_1^2 + \frac{1}{2} x_2^2 - \frac{1}{r}, \quad r = \sqrt{x_1^2 + x_2^2}. \quad (3)$$

There are arc-solutions of two types:

1. arc-solutions of the first type are arcs of epicycloids, which pass twice through the origin; Hénon proposed to denote such arcs with symbols $\pm j$, $j \in \mathbb{N}$;
2. arc-solutions of the second type which are periodic solutions just passing through the origin; such arcs are denoted by symbols i and e .

As far as all arc-solutions pass through the origin so it is possible to compose the infinite number of sequences from arc-solutions $\pm j$, $j \in \mathbb{N}$, i , e by matching these arcs with hyperbolas of two types (see [16]). Hyperbolas of the first type have their pericenters near OX axis and eccentricity $e \approx 1$, hyperbolas of the second type have pericenters near OY axis and eccentricity $e \gg 1$.

M. Hénon stated in [15] that for Newtonian potential of attraction there are two pairs of arc-solutions, namely, ii and ee , which can not be matched to each other by hyperbolas described above.

Proposition 2 (M. Hénon [4, 15]) *A sequence of arc-solutions which does not contain two identical arcs of the second type in succession is a generating solution and it is called generating sequence for the Hill problem.*

Numerical analysis of all known families of periodic solutions of the Hill problem allows to state the following

of the families are used the same as in M. Hénon's papers [9, 15]. The linked families of the Anti-Hill problem have the same name with \sim (tilde) sign above. The common orbits of two families is denoted by small circle. The same circle is used to denote the orbit at which the family reaches the extremum on Jacobi integral C . To simplify the drawing we show only small part of all known at that moment families. More detail description of connection between families of periodic orbits of Hill and Anti-Hill problem is given in the recent paper [19].

Hill problem like any non integrable problem with closed invariant subset of the phase space possesses infinite number of periodic solutions grouped into corresponding families. If one would consider Hill problem solely then a lot of families are isolated, i. e. they do not share any orbits with other families. We proposed quite natural generalization of Hill problem which demonstrates that all known for us families of periodic orbits of Hill problem connected to each other by both common generating sequences and by sharing common orbits with other families either from Hill problem or from Anti-Hill problem. It means that one can start from any periodic solution and by continuation one can reach any other periodic solution of Hill problem.

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